Optimization

PROBLEM SOLVING STRATEGY: Optimization

Show/Hide Strategy

The strategy consists of two Big Stages. The first does not involve Calculus at all; the second is identical to what you did for max/min problems.

Stage I: Develop the function.

Your first job is to develop a function that represents the quantity you want to optimize. It can depend on only *one* variable. The steps:

1. **Draw a picture** of the physical situation.
   Also note any physical restrictions determined by the physical situation.

2. **Write an equation** that relates the quantity you want to optimize in terms

3. If necessary, use other given information to **rewrite your equation in terms of a single variable.**

Stage II: Maximize or minimize the function.

You now have a standard max/min problem to solve.

4. **Take the derivative** of your equation with respect to your single variable. Then find the critical points.

5. **Determine the maxima and minima as necessary.**
Remember to **check the endpoints** if there are any.

6. **Justify your maxima or minima** either by reasoning about the physical situation, or with the first derivative test, or with the second derivative test.

7. Finally, **check to make sure you have answered the question as asked**: Re-read the problem and verify that you are providing the value(s) requested: an x or y value; or coordinates; or a maximum area; or a shortest time; whatever was asked.

[collapse]

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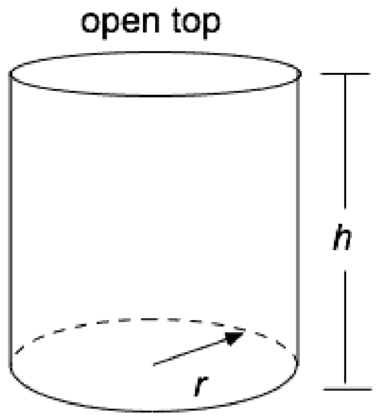
Question 1: Least expensive can

A cylindrical can, with no top lid, must contain V cm^3 of liquid. What dimensions will minimize the cost of metal to construct the can?

Show/Hide Solution

Your first job is to develop a function that represents the quantity you want to optimize. It can depend on only *one* variable. The steps: **1. Draw a picture of the physical situation.**



See the figure. We've called the radius of the cylinder r, and its height h.

2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.

We want to minimize the amount of metal we use, which is to say we want to minimize the area of the can. The can consists of a cylinder of surface area $A_{\text{cylinder}} = (2\pi r)(h)$,

and a bottom piece that has area $A_{\text{bottom}} = \pi r^2$. The can has no top. Its total area is thus:

$$\begin{aligned} A &= A_{\text{cylinder}} + A_{\text{bottom}} \\ &= 2\pi rh + \pi r^2 \end{aligned}$$

3. If necessary, use other given information to rewrite your equation in terms of a single variable.

The can's area A currently depends on two variables, r and h. In order to proceed, we must use other information we're given to rewrite the area in terms of just *one* of those variables. Let's choose r as that single variable. We must then eliminate the height h as a variable. To do so, we note that the can must hold volume V of liquid. Since the volume of a cylinder of radius r and height h is

$$V = \pi r^2 h$$

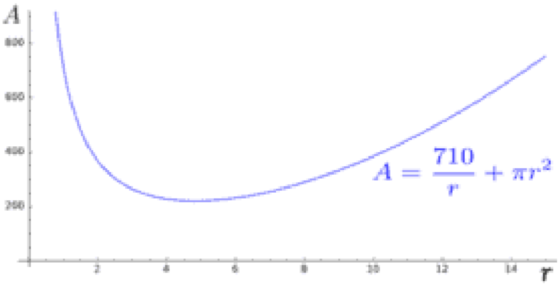
We can therefore solve for h:

$$h = \frac{V}{\pi r^2}$$

Substituting this expression for h into the expression for the can's surface area:

$$
\begin{aligned}
A &= 2\pi r h + \pi r^2 \\
&= 2\pi r\left(\frac{V}{\pi r^2}\right) + \pi r^2 \\
&= 2\cancel{\pi}\cancel{r}\left(\frac{V}{\cancel{\pi} r \cancel{^2}}\right) + \pi r^2 \\
&= \frac{2V}{r} + \pi r^2
\end{aligned}
$$

The expression for A is now a function of the single variable r. (Remember that V is a constant that just tells us how much liquid the can must hold.) We've graphed the function (with $V = 355\text{ cm}^3$), a step you probably wouldn't do yourself — but we want to emphasize that everything you've done so far is to create a function that you're now going to minimize. One choice would be to closely examine the graph to determine the value of r that minimizes A . . . but instead we're going to use the max/min techniques you learned recently!



Stage II: Maximize or minimize the function.

You now have a standard max/min problem to solve. **4. Take the derivative of your equation with respect to your single variable. Then find the critical points.**

$$
\begin{aligned}
\frac{dA}{dr} &= \frac{d}{dr}\left[\frac{2V}{r} + \pi r^2\right] \\
&= -\frac{2V}{r^2} + 2\pi r
\end{aligned}
$$

The critical points occur when $\dfrac{dA}{dr} = 0$:

$$\frac{dA}{dr} = 0 = -\frac{2V}{r^2} + 2\pi r$$

$$\frac{2V}{r^2} = 2\pi r$$
$$r^3 = \frac{V}{\pi}$$

$$r = \sqrt[3]{\frac{V}{\pi}}$$

5. Justify your maxima or minima either by reasoning about the physical situation, or with the first derivative test, or with the second derivative test.

Let's examine the second derivative. Above we found the first derivative:

$$\frac{dA}{dr} = -\frac{2V}{r^2} + 2\pi r$$

So

$$\frac{d^2A}{dr^2} = \frac{d}{dr}\left(-\frac{2V}{r^2} + 2\pi r\right)$$

$$= -2\left(-\frac{2V}{r^3}\right) + 2\pi$$

$$= \frac{4V}{r^3} + 2\pi$$

Since $r > 0$, this second derivative $\left(\frac{d^2A}{dr^2} = \frac{4V}{r^3} + 2\pi\right)$ is always positive $\left(\frac{d^2A}{dr^2} > 0\right)$. Hence this single critical point gives us a minimum:

The minimum area occurs when $r = \sqrt[3]{\frac{V}{\pi}}$.

6. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.

Recall that we found above $h = \frac{V}{\pi r^2}$. Hence when $r = \sqrt[3]{\frac{V}{\pi}}$, the cylinder's height is

$$h = \frac{V}{\pi r^2}$$

$$= \frac{V}{\pi\left(\sqrt[3]{\frac{V}{\pi}}\right)^2}$$

$$= \frac{V}{\pi}\frac{\pi^{2/3}}{V^{2/3}}$$

$$= \sqrt[3]{\frac{V}{\pi}}$$

Hence we obtain the minimum area for this open-topped can when the cylinder's height equals its radius, and the dimensions are:

$$h = r = \sqrt[3]{\frac{V}{\pi}} \quad \checkmark$$

7. Finally, check to make sure you have answered the question as asked: x or y values, or coordinates, or a maximum area, or a shortest time, or . . . .

The question asked us to specify the cylinder's dimensions, which we have provided. ✓

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[hide solution]

A poster must have a printed area of 320 cm^2. It will have top and bottom margins that are 5 cm each, and side margins that are 4 cm. What are the dimensions of the poster with the smallest total area?

Show/Hide Solution

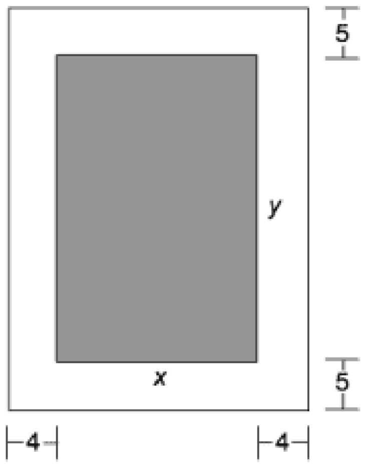
Answer: Width = 24 cm, and length =30 cm

Stage I: Develop the function.

Your first job is to develop a function that represents the quantity you want to optimize. It can depend on only *one* variable. The steps: **1. Draw a picture of the physical situation.**

See the figure. We've called the width of the printed area *x*, and its length *y*. We can then write the printed area as

$$A_{\text{printed}} = xy = 320 \text{ cm}^2$$



$A_{\text{printed}} = xy = 320 \text{ cm}^2$

Note that this picture captures the key features of the situation, and we wouldn't make good progress without it! **2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.**

We see from the figure that the poster has total width $w = x + 8$ cm, and total length $l = y + 10$ cm. Its total area is thus

$$A_{\text{total}} = (x + 8)(y + 10)$$

3. If necessary, use other given information to rewrite your equation in terms of a single variable.

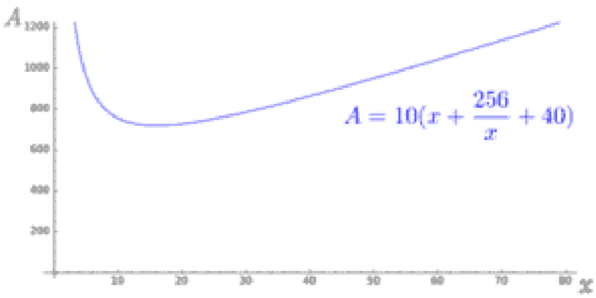
The poster's area A_{total} currently depends on two variables, *x* and *y*. In order to proceed, we must use other information we're given to rewrite the area in terms of just *one* of those variables. Let's choose the width *x* as that single variable. We must then eliminate the height *y* as a variable. To do so, recall that

Hence we can write y in terms of x as

$$y = \frac{320}{x}$$

Substituting this expression for y into our expression for the poster's total area A_{total} gives:

$$\begin{aligned}A_{\text{total}} &= (x+8)(y+10) \\ &= (x+8)\left(\frac{320}{x}+10\right) \\ &= 10(x+8)\left(\frac{32}{x}+1\right) \\ &= 10\left(32 + x + \frac{256}{x} + 8\right) \\ &= 10\left(x + \frac{256}{x} + 40\right)\end{aligned}$$



We've graphed the function, a step you probably wouldn't do yourself — but we want to emphasize that everything you've done so far is to create a function that you're now going to minimize. One choice would be to closely examine the graph to determine the value of x that minimizes A . . . but instead we're going to use the max/min techniques you learned recently!

Stage II: Maximize or minimize the function.

You now have a standard max/min problem to solve. **4. Take the derivative of your equation with respect to your single variable. Then find the critical points.**

$$\begin{aligned}\frac{dA_{\text{total}}}{dx} &= \frac{d}{dx}\left[10\left(x + \frac{256}{x} + 40\right)\right] \\ &= 10\frac{d}{dx}\left[\left(x + \frac{256}{x} + 40\right)\right]\end{aligned}$$

$$\left[\quad 256\right]$$

The critical point occurs when $\frac{dA_{\text{total}}}{dx} = 0$:

$$\frac{dA_{\text{total}}}{dx} = 0 = 10\left[1 – \frac{256}{x^2}\right]$$

$$0 = 1 – \frac{256}{x^2}$$

$$x^2 = 256$$

$$x = \sqrt{256} = 16 \text{ cm}$$

5. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.

Recall that we found above that $y = \frac{320}{x}$. Hence hen the printed area's width is $x = 16$ cm, its height is

$$y = \frac{320}{16} = 20 \text{ cm}$$

6. Justify your maxima or minima either by reasoning about the physical situation, or with the first derivative test, or with the second derivative test.

Let's use the second derivative test to verify that the critical point we've found represents a minimum. We found above that the first derivative is

$$\frac{dA_{\text{total}}}{dx} = 10\left(1 – \frac{256}{x^2}\right)$$

The second derivative is thus

$$\frac{d^2A_{\text{total}}}{dx^2} = 10\frac{d}{dx}(1) – 10(256)\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$= 0 – (-2)(10)(256)\frac{1}{x^3}$$

$$= \frac{(2)(10)(256)}{x^3}$$

Since the width is always positive ($x > 0$), this second derivative is always positive, and so our single critical point gives us a *minimum* value for the

. .

Careful! We've found x and y for the poster's printed area, but the question asks for the dimensions of the overall poster. Those values are

$$\text{width} = x + 8 = 16 + 8 = 24 \text{ cm}$$

$$\text{length} = y + 10 = 20 + 10 = 30 \text{ cm}$$

The minimum dimensions of the poster are thus width $x = 24$ cm and length $y = 30$ cm. ✓

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[hide solution]

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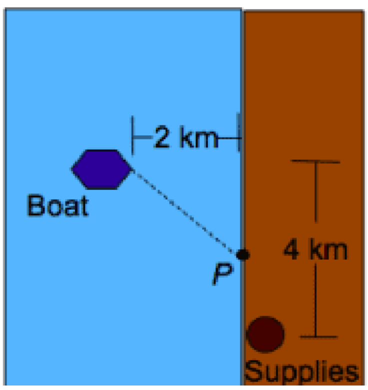
Question 3: Shortest time to row, then run, to a point on shore

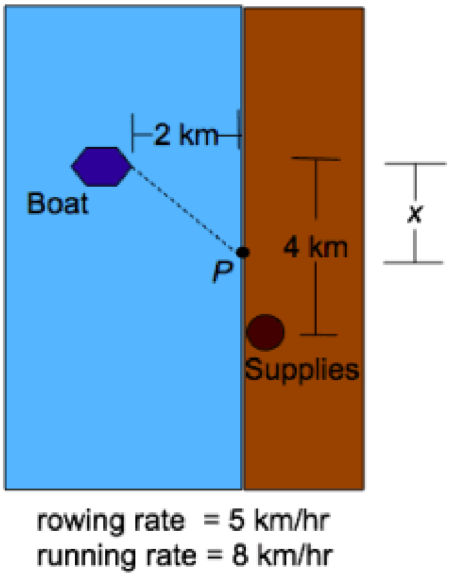
You are in a boat on a still lake, near a straight section of shore. The nearest point to shore is 2 km away. You are aiming to get back to your food and drink supplies, 4 km south of that nearest point. You can row at 5 km/hr, and run at 8 km/hr. At what point P should you land your boat to reach your supplies as quickly as



Show/Hide Solution

Answer: 1.6 km south

1. Draw a picture of the physical situation.
See the figure. We've called the distance from the point nearest on shore down to where will land the boat x.



2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.
We want to minimize the time it will take us to reach the supplies. Recall the fundamental relation

$$time = \frac{distance}{rate}$$

For instance, if we were to row straight to shore at 5 km/hr, traversing 2 km, that would take $\dfrac{2 \text{ km/hr}}{5 \text{ km/hr}} = 0.40$ hr. And then running south the 4 km at 8 km/hr would take $\dfrac{4 \text{ km/hr}}{8 \text{ km/hr}} = 0.50$ hr, for a total travel time of 0.9 hours. We can shorten this time by doing more rowing and landing at point P further south.

Or you could instead row straight to your supplies, which would take time $\dfrac{\sqrt{2^2+4^2}}{5} = 0.89$ hr.

Let's use our usual approach to find a minimum time required if we put the point P somewhere between those two extremes.

By landing at point P, we row a distance (by the Pythagorean Theorem)

$$\text{rowing time} = \frac{\text{rowing distance}}{\text{rowing rate}} = \frac{\sqrt{4+x^2}}{5}$$

We then run a distance $(4-x)$ km, at a rate of 8 km/hr, thus taking time

$$\text{running time} = \frac{\text{running distance}}{\text{running rate}} = \frac{4-x}{8}$$

The total time is thus given by the function $f(x)$

$$f(x) = \frac{\sqrt{4+x^2}}{5} + \frac{4-x}{8}$$

4. Take the derivative of your equation with respect to your single variable. Then find the critical points.

$$\frac{df}{dx} = \frac{d}{dx}\left[\frac{\sqrt{4+x^2}}{5} + \frac{4-x}{8}\right]$$

$$= \frac{1}{2}\frac{1}{5\sqrt{4+x^2}}(2x) - \frac{1}{8}$$

$$= \frac{x}{5\sqrt{4+x^2}} - \frac{1}{8}$$

The minimum time occurs when $\frac{df}{dx} = 0$:

$$\frac{df}{dx} = 0 = \frac{x}{5\sqrt{4+x^2}} - \frac{1}{8}$$

$$\frac{x}{5\sqrt{4+x^2}} = \frac{1}{8}$$

$$8x = 5\sqrt{4+x^2}$$

$$64x^2 = 25(4+x^2)$$

$$64x^2 = 100 + 25x^2$$

$$39x^2 = 100$$

5. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.
6. Justify your maxima or minima either by reasoning about the physical situation, or with the first derivative test, or with the second derivative test.
Just for completeness, let's compute the time required when you land 1.6 km south of your current location:

$$\begin{aligned} \text{total time} &= \frac{\sqrt{4+x^2}}{5} + \frac{4-x}{8} \\ &= \frac{\sqrt{4+(1.6)^2}}{5} + \frac{4-1.6}{8} \\ &= 0.81 \text{ hr} \end{aligned}$$

This time is less than the two times we computed earlier (0.9 hr if you row straight to shore; 0.89 hr if you row straight to your supplies).

7. Finally, check to make sure you have answered the question as asked: x or y values, or coordinates, or a maximum area, or a shortest time, or . . . .
The best location to land is a distance $x = 1.6$ km south, in order to minimize the time required to reach your supplies. ✓

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Question 4: Sailing ships

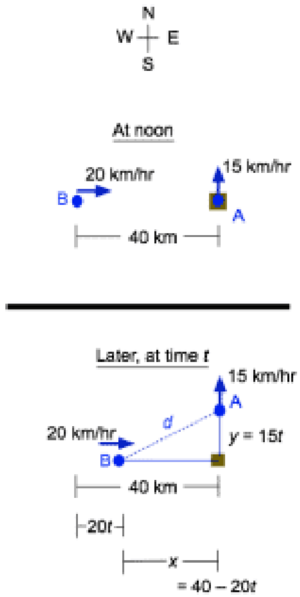
Boat A leaves a dock at noon and travels due north at 15 km/hr. Simultaneously, Boat B is heading due east toward the same dock at 20 km/hr and reaches there at 2:00 PM. At what time are the two boats closest together?

Show/Hide Solution

Answer: 1.28 hr

1. Draw a picture of the physical situation.

Once again, a good picture is key to proceeding correctly. Here we'll actually sketch two figures to represent the situation at noon, when the scenario begins, and one for some later, general time t.



The upper figure shows the situation at noon: Boat A is leaving the dock (shown as a brown square), traveling northward. Boat B is headed straight eastward toward the dock. At this instant, it is 40 km away, which we know since it will reach the dock 2 hours later (at 2:00 PM) and is traveling at 20 km/hr.

The lower figure shows the situation at later time t, where t is measured in hours. Since Boat A is traveling at 15 km/hr, it has traveled northward a distance $y = 15t$ northward of the dock.

Since Boat B is traveling at 20 km/hr, it has traveled a distance $20t$, meaning it is now a distance $x = 40–20t$ away from the dock.

The distance d between the boats at time t is given by the Pythagorean theorem:

$$\begin{aligned} d^2 &= x^2 + y^2 \\ &= (40\text{–}20t)^2 + (15t)^2 \end{aligned}$$

4. Take the derivative of your equation with respect to your single variable. Then find the critical points.

The minimum value of d occurs at the same instant as when d^2 is minimized, so let's define the function $f(t) = d^2 = (40\text{–}20t)^2 + (15t)^2$ and minimize f.

(Otherwise we would write d as the square root of everything on the right-hand side of the equation, but that just introduces a messy square root we then have to work with.)

$$\begin{aligned} \frac{df}{dt} &= \frac{d}{dt}\left[(40\text{–}20t)^2 + (15t)^2\right] \\ &= 2(40\text{–}20t)(-20) + 2(15t)(15) \\ &= (40\text{–}20t)(-40) + (30t)(15) \\ &= 40(-40) - 20t(-40) + (30)(15t) \\ &= -1600 + 800t + 450t \\ &= -1600 + 1250t \end{aligned}$$

Hence the critical point occurs for t such that

$$\frac{df}{dt} = 0 = -1600 + 1250t$$

$$1250t = 1600$$

$$t = \frac{1600}{1250} = 1.28 \text{ hr} \quad \checkmark$$

5. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.

6. Justify your maxima or minima either by reasoning about the physical situation, or with the first derivative test, or with the second derivative test.

The second derivative of f is positive ($f''(t) = 1250$), and so we know this single critical point is gives the absolute minimum.

7. Finally, check to make sure you have answered the question as asked: x or y values, or coordinates, or a maximum area, or a shortest time, or . . . .

We have found the time at which the boats are closest, as requested. ✓

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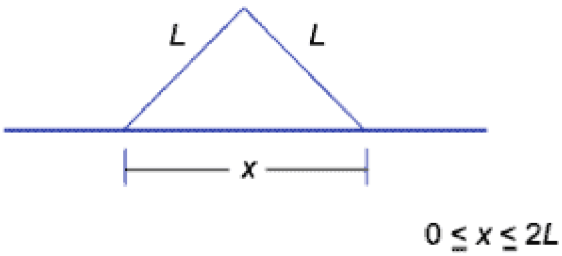
[hide solution]

You want to build a triangular enclosure with a fence on two sides and a wall on the third side. Each fence segment has length L. What is the length x of the third side (along the wall) if the region enclosed has the largest possible area? Explain your reasoning.

Show/Hide Solution

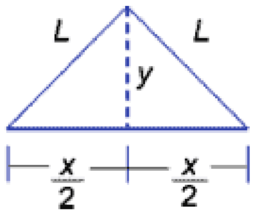
Answer: $x = \sqrt{2}L$

1. Draw a picture of the physical situation. Also note any physical restrictions determined by the physical situation.



See the top figure.

Given the physical situation, the smallest x can be is zero, when the two fences are perpendicular to the wall—in which case they enclose no area at all. The largest x can be is $2L$, if the fences both lie parallel to the wall—in which case they again enclose no area at all.



2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.

We want to maximize the area A of the triangle by choosing x appropriately. Because we remember a triangle's area as $A = \frac{1}{2}(base)(height)$, we have drawn the lower figure showing the triangle's height y. Then the area is given by

$$A = \frac{1}{2}xy$$

3. If necessary, use other given information to rewrite your equation in terms of

eliminate y from the expression. To do so, note that by the Pythagorean theorem we know:

$$\left(\frac{x}{2}\right)^2 + y^2 = L^2$$
$$y = \sqrt{L^2 - \frac{x^2}{4}}$$

Hence we can write the area in terms of the single variable x as

$$A = \frac{1}{2}x\sqrt{L^2 - \frac{x^2}{4}}$$

4. Take the derivative of your equation with respect to your single variable. Then find the critical points.

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{2}x\sqrt{L^2 - \frac{x^2}{4}}\right]$$

$$= \frac{1}{2}\frac{d}{dx}\left[x\sqrt{L^2 - \frac{x^2}{4}}\right]$$

$$= \frac{1}{2}\left[\sqrt{L^2 - \frac{x^2}{4}} + \frac{1}{2}x\frac{1}{\sqrt{L^2 - \frac{x^2}{4}}}\left(\frac{-2x}{4}\right)\right]$$

$$= \frac{1}{2}\left[\frac{L^2 - \frac{x^2}{4}}{\sqrt{L^2 - \frac{x^2}{4}}} - \frac{1}{4}x^2\frac{1}{\sqrt{L^2 - \frac{x^2}{4}}}\right]$$

$$= \frac{1}{2}\frac{1}{\sqrt{L^2 - \frac{x^2}{4}}}\left[\left(L^2 - \frac{x^2}{4}\right) - \frac{x^2}{4}\right]$$

$$= \frac{1}{2}\frac{1}{\sqrt{L^2 - \frac{x^2}{4}}}\left[L^2 - \frac{x^2}{2}\right]$$

5. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.

To find the critical point(s), we determine the values of x for which (I) $\frac{dA}{dx}$ is undefined, or (II) $\frac{dA}{dx} = 0$:

(I) $\frac{dA}{dx}$ is undefined if the expression $\sqrt{L^2 - \frac{x^2}{4}} = 0$ (since it appears in the denominator). But that occurs when $x = 2L$, and we already decided above that in that scenario the fences lie parallel to the wall, and so enclose zero area. This critical point thus represents a minimum area, rather than a maximum.

(II) When $\frac{dA}{dx} = 0$:

$$\frac{dA}{dx} = \frac{1}{2}\frac{1}{\sqrt{L^2 - \frac{x^2}{4}}}\left[L^2 - \frac{x^2}{2}\right] = 0$$

$$L^2 - \frac{x^2}{2} = 0$$

$$L^2 = \frac{x^2}{2}$$

$$x = \sqrt{2}L \quad \checkmark$$

6. Justify your maxima or minima either by reasoning about the physical situation, or with the first derivative test, or with the second derivative test.

We noted above in part (a) that the endpoints ($x = 0$ and $x = 2L$) both correspond to the minimum area $A = 0$.

We can use the first derivative test to show that our result $x = \sqrt{2}L$ corresponds to a maximum: For $x < \sqrt{2}L$, $\frac{dA}{dx} > 0$, whereas for $x > \sqrt{2}L$, $\frac{dA}{dx} < 0$. We have thus found the value of x to maximize the area.

Finally, check to make sure you have answered the question as asked: x or y values, or coordinates, or a maximum area, or a shortest time, or . . . .

The question asked for the length x, which we have provided. ✓

[hide solution]

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Question 6: Rectangle inscribed in ellipse

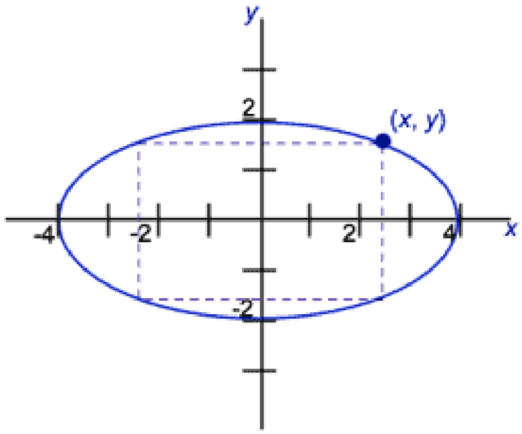
What is the area of the largest rectangle that can be inscribed in the ellipse $x^2 + 4y^2 = 16$? (Do not use a calculator; you may leave square roots in your answer.)

Show/Hide Solution

Answer: 16

1. Draw a picture of the physical situation.

See the figure. We've marked the upper right corner of the rectangle with coordinates (x, y), where x and y are related by the ellipse's equation since the rectangle's corners lie on the ellipse. The rectangle then has width $w = 2x$, and height $h = 2y$.



2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.

We want to maximize the rectangle's area, given the constraint that its corners must lie on the ellipse.

$$A = (2x)(2y)$$

a single variable.

We choose to write the ellipse's area solely in terms of x, and so we must eliminate y as a variable. To do so, note that the rectangle's corners lie on the ellipse, and so those x and y corner values are related by

$$4y^2 = 16 – x^2$$

$$y = \pm\sqrt{\frac{16 - x^2}{4}}$$

$$= \pm\frac{\sqrt{16 - x^2}}{2}$$

The positive square root corresponds to the top half of the ellipse (positive values of y), while the negative square root corresponds to the lower half of the ellipse (negative values of y). When we wrote the rectangle's height as $h = 2y$ we already assumed that y is positive, so we'll just consider the positive square root in the following analysis.

We can now write the rectangle's area as

$$A = (2x)\left(2\frac{\sqrt{16 - x^2}}{2}\right) = 2x\sqrt{16 - x^2}$$

4. Take the derivative of your equation with respect to your single variable. Then find the critical points.

$$\frac{dA}{dx} = \frac{d}{dx}\left[2x\sqrt{16-x^2}\right]$$

$$= 2\left[\sqrt{16-x^2} + x\,\frac{1}{2}\,\frac{-2x}{\sqrt{16-x^2}}\right]$$

$$= 2\left[\frac{16-x^2}{\sqrt{16-x^2}} - \frac{x^2}{\sqrt{16-x^2}}\right]$$

$$= 2\frac{16-2x^2}{\sqrt{16-x^2}}$$

5. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.

The critical points occur when (I) $\frac{dA}{dx}$ is undefined, or (II) $\frac{dA}{dx} = 0$.

(I) $\frac{dA}{dx}$ is undefined when the denominator is zero, or when $16 - x^2 = 0$, and so when $x = \pm 4$. Note, however, that those are the endpoints of the ellipse, and so $y = 0$ there and the rectangle has no area. This critical point thus gives a minimum rather than a maximum area.

(II)When $\frac{dA}{dx} = 0$:

$$16 - 2x^2 = 0$$
$$x^2 = 8$$
$$x = \pm\sqrt{8} = \pm 2\sqrt{2}$$

6. Justify your maxima or minima either by reasoning about the physical situation, or with the first derivative test, or with the second derivative test.

we see that at our positive critical point $x = 2\sqrt{2}$ (or, equivalently, $x^2 = 8$), $\dfrac{dA}{dx}$ goes from positive to negative, and hence A is a maximum there.

Finally, check to make sure you have answered the question as asked: x or y values, or coordinates, or a maximum area, or a shortest time, or . . . .

Careful! The question asked for the maximum area that can be inscribed, and so to finish the problem we must provide that.

$$A_{\text{max}} = 2x\sqrt{16 - x^2} \text{ with } x = \sqrt{8}$$

$$= 2\sqrt{8}\sqrt{16–8}$$

$$= 2\sqrt{8}\sqrt{8} = 2 \cdot 8 = 16 \quad ✓$$

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[hide solution]

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Question 7: Triangle with vertex on a curve

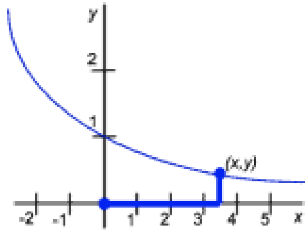
A right triangle has one vertex at the origin and one vertex on the curve $y = e^{-x/2}$ for $1 \leq x \leq 4$. One of the two perpendicular sides is along the x-axis; the other is

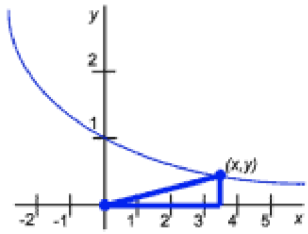
Show/Hide Solution

Answer: Minimum: 0.27; maximum: 0.37

1. Draw a picture of the physical situation.

See the upper figure. We had trouble envisioning this at first, but drawing the picture while working through the description makes it clear: One vertex is on the origin, so mark that spot first. One of the perpendicular sides is along the x-axis, so draw that next. The other perpendicular side is parallel to the y-axis, and must end on the curve, so draw that. We're showing it ending on the point (x, y) on the curve.





$1 \le x \le 4$

Finally, the third side, the hypotenuse, must then go from that point back to the origin. See the lower figure.

2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.

The triangle has area $A = \frac{1}{2}(base)(height)$:

$$A = \frac{1}{2}xy$$

3. If necessary, use other given information to rewrite your equation in terms of a single variable.

We choose to write the area solely in terms of the variable x; hence we must eliminate y. To do so, note that since the point (x, y) lies on the curve, y is related to x by $y = e^{-x/2}$. Hence

$$A = \frac{1}{2}xe^{-x/2}$$

4. Take the derivative of your equation with respect to your single variable. Then find the critical points.

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{2}xe^{-x/2}\right]$$

$$= \frac{1}{2}\left[e^{-x/2} + x\left(-\frac{1}{2}\right)e^{-x/2}\right]$$

$$= \frac{e^{-x/2}}{2}\left[1 - \frac{1}{2}x\right]$$

The only critical point, when $\frac{dA}{dt} = 0$, occurs when $x = 2$.

5. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.

We now just need to check the critical point ($x = 2$), and the endpoints ($x = 1$ and $x = 4$), to find the minimum and maximum areas:

$$A = \frac{1}{2}xe^{-x/2}$$

For $x = 1$: $$A = \frac{1}{2}(1)e^{-1/2} \approx 0.30$$

For $x = 2$: $$A = \frac{1}{2}(2)e^{-1} \approx 0.37$$

For $x = 4$: $$A = \frac{1}{2}(4)e^{-2} \approx 0.27$$

7. Finally, check to make sure you have answered the question as asked: x or y values, or coordinates, or a maximum area, or a shortest time, or . . . .

The question asks for the minimum and maximum areas, which we have provided. ✓

[hide solution]

↑

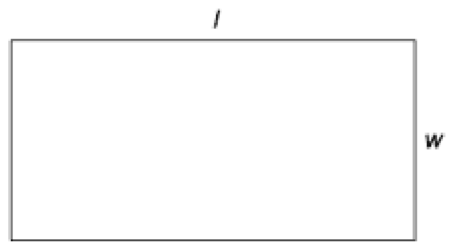
Question 8: Rectangle with the greatest area is a square

Prove that of all rectangles with given perimeter P, the square has the largest area.

Show/Hide Solution

1. Draw a picture of the physical situation.

Many students don't know where to begin when faced with this question. As always, we start with a figure: we're trying to demonstrate something about a rectangle, so we draw a rectangle, with sides labeled w and l for the width and length. See



The rectangle has a given perimeter P, so

$$2w + 2l = P$$

2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.

We want to maximize the area of the rectangle,

$$A = lw$$

3. If necessary, use other given information to rewrite your equation in terms of a single variable.

We choose to write the area solely in terms of the width w, and so we need to eliminate the length l as a variable. To do so, note that we can rewrite the length in terms of the width and the given perimeter P:

$$\begin{aligned} 2w + 2l &= P \\ l &= \frac{1}{2}P - w \end{aligned}$$

Thus we can write the area as

$$A = \left(\frac{1}{2}P - w\right)w$$

$$= \frac{1}{2}Pw - w^2$$

3. Take the derivative of your equation with respect to your single variable. Then find the critical points.

Remember that the perimeter P is a given (it could be, say, 10 cm, or 23 inches;

maximize the rectangle's area, so we take the derivative with respect to w:

$$\frac{dA}{dw} = \frac{d}{dw}\left[\frac{1}{2}Pw – w^2\right]$$

$$= \frac{1}{2}P – 2w$$

The critical points occur when $\frac{dA}{dw} = 0$:

$$\frac{dA}{dw} = 0 = \frac{1}{2}P – 2w$$

$$2w = \frac{1}{2}P$$

$$w = \frac{1}{4}P$$

5. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.

6. Justify your maxima or minima either by reasoning about the physical situation, or with the first derivative test, or with the second derivative test.

Since $\frac{dA}{dw} = \frac{1}{2}P – 2w$, as we increase the width from zero, $\frac{dA}{dw}$ goes from positive to negative at $w = \frac{1}{4}P$, and so by the first derivative test this value of w corresponds to the *maximum* area for the rectangle.

When $w = \frac{1}{4}P$,

$$l = \frac{1}{2}P – w$$

$$= \frac{1}{2}P – \frac{1}{4}P = \frac{1}{4}P$$

and length are equal to each other, and to one-quarter the perimeter of the total perimeter: $w = l = \frac{1}{4}P$. That is, the area is maximized when the rectangle has equal sides, and so is a square. ✓

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[hide solution]

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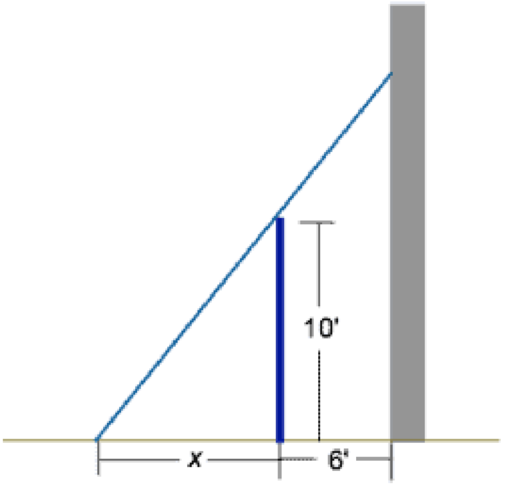
Question 9: Shortest ladder required to reach a house over a wall

A wall 10 feet high is six feet from a house. Find the length of the shortest ladder that will reach the house while leaning against the fence.

[*Hint:* The math works out more easily if you call x the distance from the base of the ladder to the wall (rather than to the house); see the figure. Even so, the algebra at the end can get messy. Don't spend more than a few minutes on that if you can't easily find the critical point(s); instead just check your work against ours up to that point, and then see a convenient way to factor.]

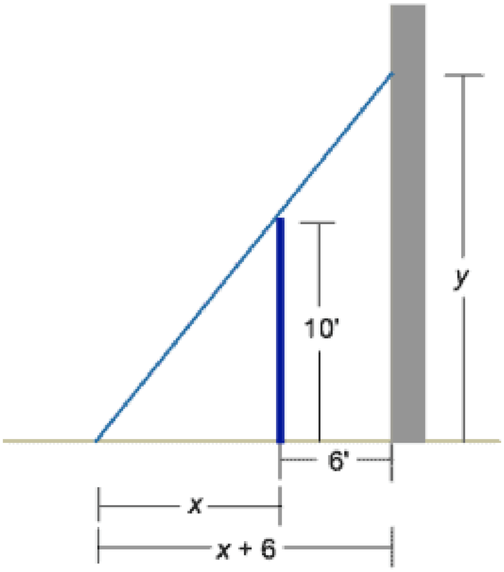


Show/Hide Solution

Answer: 22.39 feet

1. Draw a picture of the physical situation.

See the figure. As suggested by the hint, we've called the distance from the ladder's base to the wall x, so the horizontal distance from the ladder's base to the house is $x + 6$. We're calling the ladder's height y.



2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.

The ladder's length L is given by

$$L^2 = (x + 6)^2 + y^2$$

3. If necessary, use other given information to rewrite your equation in terms of a single variable.

We choose to write the ladder's length solely in terms of the variable x, and so we need to eliminate y. To do so, note that by similar triangles:

$$\frac{y}{10} = \frac{x + 6}{x}$$

$$y = (10)\frac{x + 6}{x} = (10)\left(1 + \frac{6}{x}\right)$$

We can then rewrite our expression for L solely in terms of x:

$$L^2 = (x + 6)^2 + (10)^2\left(1 + \frac{6}{x}\right)^2$$

4. Take the derivative of your equation with respect to your single variable. Then find the critical points.

The minimum value of L occurs for the same value of x as when L^2 is minimized, so let's define the function $f(x) = L^2 = (x+6)^2 + 100\left(1+\frac{6}{x}\right)^2$

and minimize f. (Otherwise we would write L as the square root of everything on the right-hand side of the equation, but that just introduces a messy square root we then have to work with.)

$$\frac{df}{dx} = \frac{d}{dx}\left[(x+6)^2 + 100\left(1+\frac{6}{x}\right)^2\right]$$

$$= 2(x+6) + 2(100)\left(1+\frac{6}{x}\right)\left(\frac{-6}{x^2}\right)$$

$$= 2(x+6) – 2(600)\left(\frac{x+6}{x}\right)\left(\frac{1}{x^2}\right)$$

$$= 2(x+6)\left[1 – \left(\frac{600}{x^3}\right)\right]$$

The minima occur when $\frac{df}{dx} = 0$:

$$\frac{df}{dx} = 0 = 2(x+6)\left[1 – \left(\frac{600}{x^3}\right)\right]$$

$x = -6$ is physically impossible. Hence our single critical point is

$x = \sqrt[3]{600} = 8.43 \text{ ft}.$

5. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.

6. Justify your maxima or minima either by reasoning about the physical

say, $x = 10$). Hence by the first derivative test $x = \sqrt[3]{600}$ gives us a minimum for f, and hence for L^2.

7. Finally, check to make sure you have answered the question as asked: x or y values, or coordinates, or a maximum area, or a shortest time, or . . . .

Careful! The question asked for the ladder's shortest length, so to finish we must compute that.

When $x = \sqrt[3]{600} = 8.43$ ft,

$$L^2 = (x+6)^2 + 100\left(1 + \frac{6}{x}\right)^2$$

$$= (8.43+6)^2 + 100\left(1 + \frac{6}{8.43}\right)^2$$

$$= 501.23$$

$$L = \sqrt{501.23} = 22.39 \text{ ft} \quad \checkmark$$

[hide solution]

A man has 340 yards of fencing to enclose two separate fields. One field will be a rectangle twice as long as it is wide, and must contain at least 800 square yards. The other field is a square and must contain at least 100 square yards.

(a) If x is the width of the rectangular field, what are the maximum and minimum possible values of x?

(b) What is the greatest number of square yards that can be enclosed by the two fields? Justify your answer.

Show/Hide Solution

Solution Summary   Solution (a) Detail   Solution (b) Detail

(a) 20 yards $\leq x \leq$ 50 yards

(b) 5100 square yards

[hide solution]

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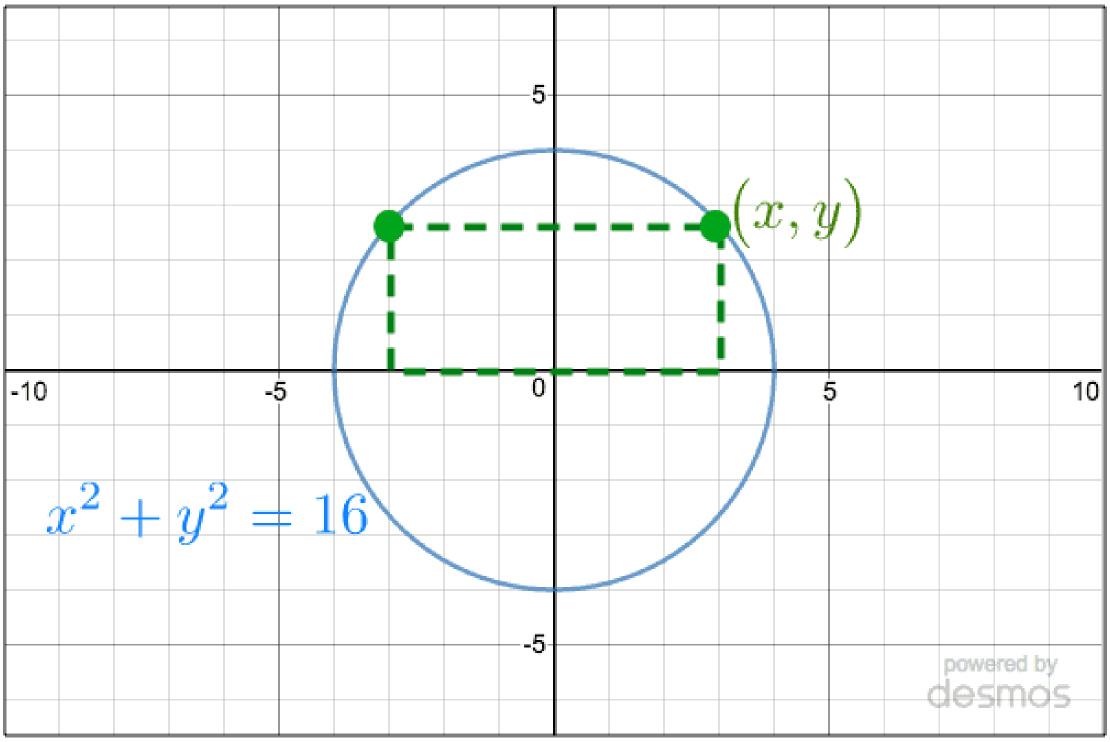
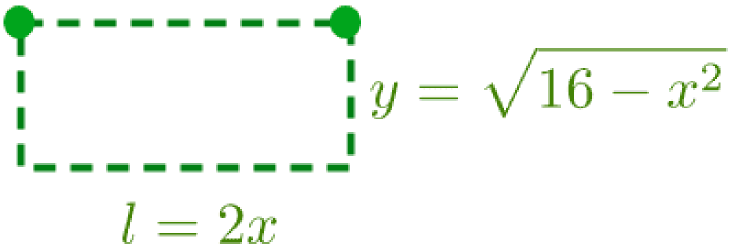
Question 11: Rectangle in circle

Show/Hide Solution

1. Draw a picture of the physical situation.



(a) A rectangle inscribed by the top half of the circle shown. Its upper right corner lies at the point $(x, y) = (x, \sqrt{16 - x^2})$.



(b) The rectangle thus has length $l = 2x$ and height $y = \sqrt{16 - x^2}$.

See the upper figure (a): the rectangle is inscribed in the upper half of the given circle. Its upper right corner lies at the point (x, y) on the circle.

2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.

As shown in the lower figure (b), the rectangle has length $l = 2x$. (The right-half has length x, and the left-half also has length x, so its total length is $2x$.) It has height y. Its area equals its length multiplied by its height, and so

$$A = 2xy$$

3. If necessary, use other given information to rewrite your equation in terms of a single variable.

Here's the key to this problem: for every point on the circle, the values of x and y are related by $x^2 + y^2 = 16$. Hence $y = \pm\sqrt{16 - x^2}$. We're interested in the point in the upper-right quadrant (as shown in the figure), and so we want the positive value of y :

$$y = \sqrt{16 - x^2}$$

We can thus rewrite the rectangle's area in terms of the single variable x :

$$\begin{aligned} A &= 2xy \\ &= 2x\sqrt{16 - x^2} \end{aligned}$$

The graph to the right shows the area A as a function of x. (Recall that the rectangle's upper right corner is marked by the point (x, y).) You would probably never graph this for yourself, but we want to emphasize that we've just determined the function $A(x)$ so that we can now work to maximize it using your usual max/min tools.

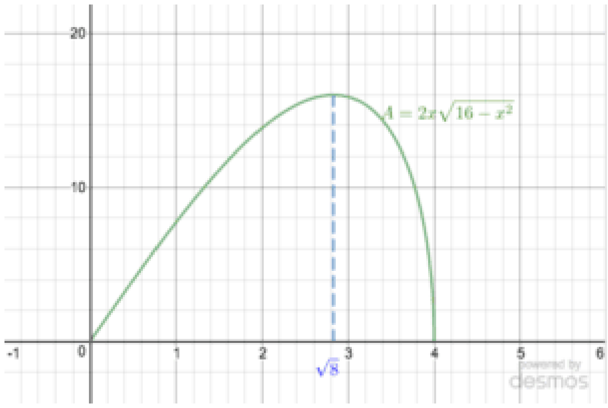
4. Take the derivative of your equation with respect to your single variable. Then find the critical points.

$$\begin{aligned}\frac{dA}{dx} &= \frac{d}{dx}\left[2x\sqrt{16-x^2}\right] \\ &= \left[\frac{d}{dx}(2x)\right]\sqrt{16-x^2} + 2x\left[\frac{d}{dx}\sqrt{16-x^2}\right] \\ &= 2\sqrt{16-x^2} + 2x\left[\frac{1}{2\sqrt{16-x^2}}\frac{d}{dx}\left(16-x^2\right)\right] \\ &= 2\sqrt{16-x^2} + 2x\left[\frac{-2x}{2\sqrt{16-x^2}}\right] \\ &= 2\sqrt{16-x^2} – \frac{2x^2}{\sqrt{16-x^2}} \\ &= \frac{2\sqrt{16-x^2}\sqrt{16-x^2} - 2x^2}{\sqrt{16-x^2}} \\ &= \frac{2\left(16-x^2\right) - 2x^2}{\sqrt{16-x^2}} \\ &= \frac{32–4x^2}{\sqrt{16-x^2}}\end{aligned}$$

Now, the critical points occur when either the numerator or the denominator equal zero. The denominator equals zero when $x = \pm 4$, but those points mark the circle's edge where it intersects with the *x*-axis, and so the rectangle would have zero height there. (Look back at the figure above.) These points therefore cannot give us a maximum area. So let's concentrate on the numerator: these critical points occur when

$$\begin{aligned}32–4x^2 &= 0 \\ 4x^2 &= 32 \\ x^2 &= 8 \\ x &= \pm\sqrt{8}\end{aligned}$$

The graph to the right shows the plot of $A(x)$ versus x again, this time with a vertical line marking where $x = \sqrt{8} \approx 2.83$, which does visually appear to



5&6. Determine the maxima and minima as necessary. Justify your maxima or minima either by reasoning about the physical situation, or with the first derivative test, or with the second derivative test.

We want to make sure that we've found a maximum for the rectangle's area using Calculus tools. Let's use the First Derivative Test:

Recall that

$$\frac{dA}{dx} = \frac{32 – 4x^2}{\sqrt{16 - x^2}}$$

The denominator is always positive, so we can ignore that and consider only the numerator $32–4x^2$. For values of x less than our critical value of $x = \sqrt{8} \approx 2.83$, the value of $\frac{dA}{dx}$ is positive; for instance, at $x = 2$, the numerator is $32–4(2)^2 = 16$, as positive value. By contrast, for values of x greater than our critical value of $x = \sqrt{8} \approx 2.83$, the value of $\frac{dA}{dx}$ is negative; for instance, at $x = 3$, the numerator is $32–4(3)^2 = -4$, a negative value.

Hence, since the first derivative $\frac{dA}{dx}$ goes from *positive* to *negative* at our critical value of $x = \sqrt{8}$, this marks a *maximum* for the function $A(x)$. ✓

7. Finally, check to make sure you have answered the question as asked: Re-read the problem and verify that you are providing the value(s) requested: an x or y value; or coordinates; or a maximum area; or a shortest time; whatever was asked.

The question asked for the *dimensions* of the largest rectangle, so we have a little work left to do: The largest rectangle has length

$$y = \sqrt{16 – (\sqrt{8})^2}$$
$$= \sqrt{16 – 8}$$
$$= \sqrt{8} \quad \checkmark$$

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[hide solution]

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Question 12: Triangle in curve

A right triangle has one vertex at the origin and one vertex on the curve $y = e^{-x/2}$ for $1 \leq x \leq 4$. One of the two perpendicular sides is along the x-axis; the other is parallel to the y-axis. Find the maximum and minimum areas for such a triangle.

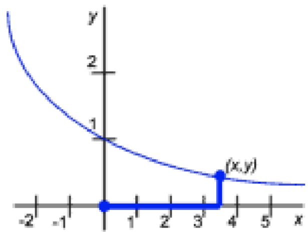
Note: You may use a calculator to compute the areas.

Show/Hide Solution

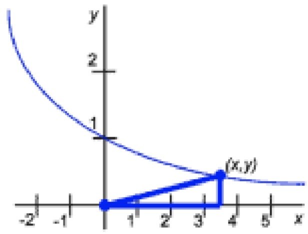
Answer: Minimum: 0.27; maximum: 0.37

1. Draw a picture of the physical situation.

See the upper figure. We had trouble envisioning this at first, but drawing the picture while working through the description makes it clear: One vertex is on and must end on the curve, so draw that. We're showing it ending on the point (x, y) on the curve.



Finally, the third side, the hypotenuse, must then go from that point back to the origin. See the lower figure.



1 ≤ x ≤ 4

2. Write an equation that relates the quantity you want to optimize in terms of the relevant variables.

The triangle has area $A = \frac{1}{2}(base)(height)$:

$$A = \frac{1}{2}xy$$

3. If necessary, use other given information to rewrite your equation in terms of a single variable.

We choose to write the area solely in terms of the variable x; hence we must eliminate y. To do so, note that since the point (x, y) lies on the curve, y is related to x by $y = e^{-x/2}$. Hence

$$A = \frac{1}{2}xe^{-x/2}$$

4. Take the derivative of your equation with respect to your single variable. Then find the critical points.

$$\frac{dA}{dx} = \frac{d}{dx}\left[\frac{1}{2}xe^{-x/2}\right]$$

$$= \frac{1}{2}\left[e^{-x/2} + x\left(-\frac{1}{2}\right)e^{-x/2}\right]$$

The only critical point, when $\dfrac{dA}{dt} = 0$, occurs when $x = 2$.

5. Determine the maxima and minima as necessary. Remember to check the endpoints if there are any.

We now just need to check the critical point ($x = 2$), and the endpoints ($x = 1$ and $x = 4$), to find the minimum and maximum areas:

$$A = \frac{1}{2}xe^{-x/2}$$

$$\text{For } x = 1: \qquad A = \frac{1}{2}(1)e^{-1/2} \approx 0.30$$

$$\text{For } x = 2: \qquad A = \frac{1}{2}(2)e^{-1} \approx 0.37$$

$$\text{For } x = 4: \qquad A = \frac{1}{2}(4)e^{-2} \approx 0.27$$

Hence the minimum area is 0.27 (when $x = 4$), and the maximum area is 0.37 (when $x = 2$). ✓

7. Finally, check to make sure you have answered the question as asked: x or y values, or coordinates, or a maximum area, or a shortest time, or . . . .

The question asks for the minimum and maximum areas, which we have provided. ✓

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- What questions do you have about the solutions above?
- Which ones are giving you the most trouble?
- What other Optimization problems are you trying to work through for your class?

If you let us know, we'll do our best to help!

As of September 2022, we’re using our Forum for comments and discussion of this topic, and for any math questions. We’d love to see you there and help! Please tap to visit **our Forum: community.matheno.com**.

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