Παράδειγμα. Επίλυση ΣΓΕ με τη μέθοδο του αντιστρόφου.

Έστω $\left\{\begin{array}{l} x+2y-z=1 \\ x+y-z=-2 \\ -2x-y+z=1 \end{array}\right\} \Rightarrow A\bar{x}=\bar{b},\ \bar{x}=(x,y,z)^T,\ \bar{b}=(1\ -2\ 1)^T$

$$A=\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \\ -2 & -1 & -1 \end{pmatrix}_{3\times 3}$$

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & -2 \\ -2 & -1 & -1 \end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}$$
$$+(-1)(-1)^{1+3}\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 4 \neq 0.$$

Άρα A μη-ιδιάζων (αντιστρέψιμος), δηλ. ΣΓΕ συμβιβαστό, οπότε $\bar{x}=A^{-1}\cdot\bar{b}$ με $A^{-1}=|A|^{-1}\,Adj(A_{ij})^T$.
Έχουμε

$$A_{11}=(-1)^{1+1}|M_{11}| = \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} = -1,\quad A_{12}=(-1)^{1+2}|M_{12}| = -\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} = 3$$

$$A_{13}=(-1)^{1+3}|M_{13}| = \begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = -3,\quad A_{21}=(-1)^{2+1}|M_{21}| = -\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = -1$$

$$A_{22}=(-1)^{2+2}|M_{22}| = \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} = -1 \quad A_{23}=(-1)^{2+3}|M_{23}| = -\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = -3.$$

$$A_{31}=(-1)^{3+1}|M_{31}| = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -3,\quad A_{32}=(-1)^{3+2}|M_{32}| = -\begin{vmatrix} 1 & -1 \\ 1 & -2 \end{vmatrix} = 1$$

$$A_{33}=(-1)^{3+3}|M_{33}| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1.$$

Άρα $Adj(A)=(A_{ij})=\begin{pmatrix} -1 & -1 & -3 \\ 3 & -1 & 1 \\ 1 & -3 & -1 \end{pmatrix}$, οπότε $A^{-1}=\frac{1}{4}\begin{pmatrix} -1 & 3 & 1 \\ -1 & -1 & -3 \\ -3 & -3 & -1 \end{pmatrix}$ και άρα

$\bar{x}=(-1,1,0)^T\ (=A^{-1}\cdot\bar{b})$.

Υπολογισμός Αντιστρόφου. Μέθοδος Επαυξημένου πίνακα. Έστω $A\in\mathbb{R}^{n\times n}$

(A) Έστω $E=(A|I_n)\sim(I_n|B)$, τότε $B=A^{-1}$.

(B) Έστω $(A|I_n)\sim(K|B)$, $K\in\mathbb{R}^{n\times n}_{rref}\neq I_n \Rightarrow A$ ιδιάζον πίνακας

Παράδειγμα. Έστω

$$A=\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0\\ 3 & 2 & 4 & 0 & 1 & 0\\ 0 & 1 & -1 & 0 & 0 & 1\end{array}\right) r_2=r_2-3r_1$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0\\ 0 & 5 & -2 & -3 & 1 & 0\\ 0 & 1 & -2 & 0 & 0 & 1\end{array}\right) \begin{array}{l} r_2:=r_3\\ r_3:=r_2\end{array} \sim\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0\\ 0 & 1 & -2 & 0 & 0 & 1\\ 0 & 5 & -2 & -3 & 1 & 0\end{array}\right)$$

$$r_1:=r_1+r_2 \sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1\\ 0 & 1 & -2 & 0 & 0 & 1\\ 0 & 5 & -2 & -3 & 1 & 0\end{array}\right) r_3:=r_3-5r_2$$

$$\sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1\\ 0 & 1 & -2 & 0 & 0 & 1\\ 0 & 0 & 8 & -3 & 1 & -5\end{array}\right) r_3:=r_3/8 \sim\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1\\ 0 & 1 & -2 & 0 & 0 & 1\\ 0 & 0 & 1 & -3/8 & 1/8 & -5/8\end{array}\right)$$

$$r_2=r_2-2r_3$$

$$\sim\left(\begin{array}{ccc} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right|\underbrace{\left.\begin{array}{ccc} 1 & 0 & 1\\ -3/4 & 1/4 & -1/4\\ -3/8 & 1/8 & -5/8\end{array}\right)}_{A^{-1}}.$$

Ασκήσεις.

(α) $\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 4 & 5 & -2 \end{pmatrix}^{-1}$ (β) $\begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 3 & 5 \end{pmatrix}$

(γ) $$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & -1 \\ a & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{1-a} \begin{pmatrix} 1 & -a & -a \\ -a & 1 & 1 \\ -a & a^2 & 1 \end{pmatrix}, \quad a \in \mathbb{R}^*$$

(δ) $$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1/2 & 2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & -1 & -1/2 \end{pmatrix}$$

(ε) $$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 5 & -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 4/3 & -1/3 \\ 3 & -7/3 & 1/3 \end{pmatrix}$$

Υπολογισμός Αντιστρόφου πίνακα. Μέθοδος του επαυξημένου πίνακα

$(A \mid I_n)$ A-επαυξημένος πίνακας $\overset{ορ.}{\iff}$ $(a_{ij} \mid I_n) = (b_{ij}) \in \mathbb{R}^{n \times 2n}$

όπου $b_{ij} := \begin{cases} a_{ij} & i,j = 1,2,\dots,n \\ \delta_{ij} & i = n+1,\dots,2n,\ j = 1,\dots,n. \end{cases}$

$(A \mid I_n)$ (με επιτρεπτές γραμμοπράξεις) $\Rightarrow (I_n \mid B)$

$\Rightarrow$ A μη-ιδιάζον με $A^{-1} = B$

Παράδειγμα. Έστω $A = \begin{pmatrix} 2 & 1 & 3 \\ 6 & 2 & 6 \\ 4 & 4 & 8 \end{pmatrix}$

Άρα $(A \mid I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 6 & 2 & 6 & 0 & 1 & 0 \\ 4 & 4 & 8 & 0 & 0 & 1 \end{array}\right)$ $\begin{array}{l} r_2 := r_2 - 3r_1 \\ r_3 := r_3 - 2r_1 \end{array}$

$= \left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 1 & 0 \\ 0 & 2 & 2 & -2 & 0 & 1 \end{array}\right)$ $\begin{array}{l} r_1 := r_1/2 \\ r_2 := -r_2 \\ r_3 := r_3 + 2r_2 \end{array}$

$= \left(\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 1 & 3 & 3 & -1 & 0 \\ 0 & 2 & 2 & -2 & 0 & 1 \end{array}\right)$ $\begin{array}{l} r_1 := r_1 + r_2/2 \\ \\ r_3 := r_3 + 2r_2 \end{array}$

$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1/2 & 0 \\ 0 & 1 & 3 & 3 & -1 & 0 \\ 0 & 0 & -4 & -8 & 2 & 1 \end{array}\right)$ $\begin{array}{l} r_2 := r_2 - \frac{3}{4} r_3 \\ r_3 := -r_3/4 \end{array}$

$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1/2 & 0 \\ 0 & 1 & 0 & -3 & 1/2 & 3/4 \\ 0 & 0 & 1 & 2 & -1/2 & -1/4 \end{array}\right) = (I_3 \mid A^{-1})$

Παράδειγμα. Εύρεση αντιστρόφου (μέθοδος επαυξημένου πίνακα ή μέθοδος στοιχειωδών μετασχηματισμών.

Έστω $A = \begin{pmatrix} 3 & -1 & 4 \\ 5 & 1 & -3 \\ 4 & -1 & 1 \end{pmatrix}$. Έχουμε.

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 3 & -1 & 4 & 1 & 0 & 0 \\ 5 & 1 & -3 & 0 & 1 & 0 \\ 4 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad r_1 := r_1/3.$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/3 & 4/3 & 1/3 & 0 & 0 \\ 5 & 1 & -3 & 0 & 1 & 0 \\ 4 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} r_2 := r_2 - 5r_1 \\ r_3 := r_3 - 4r_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1/3 & 4/3 & 1/3 & 0 & 0 \\ 0 & 8/3 & -29/3 & -5/3 & 1 & 0 \\ 0 & 1/3 & -13/3 & -4/3 & 0 & 1 \end{array}\right) \quad r_1 := r_1 + r_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 0 & 1 \\ 0 & 8/3 & -29/3 & -5/3 & 1 & 0 \\ 0 & 1/3 & -13/3 & -4/3 & 0 & 1 \end{array}\right) \quad r_2 := r_2 \cdot 3/8$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 0 & 1 \\ 0 & 1 & -29/8 & -5/8 & 3/8 & 0 \\ 0 & 1/3 & -13/3 & -4/3 & 0 & 1 \end{array}\right) \quad r_3 := r_3 - \frac{1}{3} r_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 0 & 1 \\ 0 & 1 & -29/8 & -5/8 & 3/8 & 0 \\ 0 & 0 & -25/8 & -9/8 & -1/8 & 1 \end{array}\right) \quad r_3 := r_3 \cdot \frac{8}{25}(-1)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & -1 & 0 & 1 \\ 0 & 1 & -29/8 & -5/8 & 3/8 & 0 \\ 0 & 0 & 1 & 9/25 & 1/25 & -8/25 \end{array}\right) \quad r_1 := r_1 + 3r_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & -29/8 & -5/8 & 3/8 & 0 \\ 0 & 0 & 1 & 9/25 & 1/25 & -8/25 \end{array}\right) \quad r_2 := r_2 + \frac{29}{8} r_3$$

$$\sim \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\ \middle|\ \underbrace{\begin{matrix} 2/25 & 3/25 & 1/25 \\ -5/8 & 13/8 & -29/25 \\ 9/25 & 1/25 & -8/25 \end{matrix}}_{A^{-1}}\right)$$

Παράδειγμα. Υπολογισμός αντιστρόφου (μέθοδος επαυξημένου πίνακα).

Έστω $A = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right)$ $\begin{array}{l} r_2 := r_2 - 2r_1 \\ r_3 := r_3 - r_1 \end{array}$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \quad r_3 := r_3 + 2r_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right) \quad r_3 := r_3 + 2r_2$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right) \quad r_3 := -r_3$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \quad \begin{array}{l} r_1 := r_1 - 3r_3 \\ r_2 := r_2 + 3r_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \quad r_1 := r_1 - 2r_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

Παράδειγμα. Μέθοδος επαυξημένου πίνακα εύρεσης αντιστρόφου.

Έστω $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

Έχουμε
$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \Gamma_2 := \Gamma_2 - 2\Gamma_1 \\ \Gamma_3 := \Gamma_3 - 4\Gamma_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \Gamma_2 := \Gamma_3 \\ \Gamma_3 := \Gamma_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 1 \\ 0 & -1 & -1 & -2 & 1 & 0 \end{array}\right) \begin{array}{l} \\ \\ \Gamma_3 := \Gamma_3 + \Gamma_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right) \begin{array}{l} \Gamma_1 := \Gamma_1 - 2\Gamma_3 \\ \\ \\ \end{array}$$

$$\sim \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\,\middle|\,\underbrace{\begin{matrix} -1 & 2 & 0 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{matrix}}_{A^{-1}}\right)$$

Παράδειγμα.

Έστω $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Έχουμε $(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \; r_2 := r_2 - 2r_1$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) =: (A' \mid B)$$

όπου $A' \in \mathbb{R}^{3\times 3}_{ref}$ με μηδενική γραμμή. Άρα A ιδιάζον (μη-αντιστρεψ.)

Παράδειγμα. (Α) Μέθοδος επαυξημένου πίνακα εύρεσης αντιστρόφου

Έστω $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$

Έχουμε $(A \mid I_3) = \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} r_2 := r_2 - 2r_1 \\ r_3 := r_3 + r_1 \end{array}$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right) r_3 := r_3 + r_2$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right) r_2 := r_2 + 2r_3$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) r_2 := -r_2/8$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) r_1 := r_1 - r_3$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1/2 & -3/8 & 1/4 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) r_1 := r_2 - r_1$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1/2 & -5/8 & -5/4 \\ 0 & 1 & 0 & 1/2 & -3/8 & 1/4 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right) r_1 := r_1/2 \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right).$$

(Β) Μέθοδος επισυνημμένου πίνακα εύρεσης αντιστρόφου.

Έχουμε $|A| = \begin{vmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{vmatrix}$

$$= 4(-1)^{2+1}\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} - 6(-1)^{2+2}\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} \neq 0.$$

$$A_{11} = (-1)^{1+1}|M_{11}| = \begin{vmatrix} -6 & 0 \\ 7 & 2 \end{vmatrix}$$

$$A_{12} = (-1)^{1+2}|M_{12}| = -\begin{vmatrix} 4 & 0 \\ -2 & 2 \end{vmatrix}$$

$$A_{13} = (-1)^{1+3}|M_{13}| = \begin{vmatrix} 4 & -6 \\ -2 & 7 \end{vmatrix}$$

$$A_{21} = (-1)^{2+1}|M_{21}| = -\begin{vmatrix} 1 & 1 \\ 7 & 2 \end{vmatrix}$$

$$A_{22} = (-1)^{2+2}|M_{22}| = \begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix}$$

$$A_{23} = (-1)^{2+3}|M_{23}| = -\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix}$$

$$A_{31} = (-1)^{3+1}|M_{31}| = \begin{vmatrix} 1 & 1 \\ -6 & 0 \end{vmatrix}, \quad A_{32} = (-1)^{3+2}|M_{32}| = -\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix}$$

$$A_{33} = (-1)^{3+3}|M_{33}| = \begin{vmatrix} 2 & 1 \\ 4 & -6 \end{vmatrix}$$

Άρα $A^{-1} = \frac{1}{|A|} Adj(A) = |A|^{-1}(A_{ji})$.

Παράδειγμα. Υπολογισμός αντιστρόφου (μέθοδος επαυξημένου πίνακα)

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 7 \\ 4 & -1 & a \end{pmatrix}_{3\times 3}, \quad a \in \mathbb{R}.$$

Έχουμε

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 2 & -1 & 7 & 0 & 1 & 0 \\ 4 & -1 & a & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ r_2 := r_2 - 2r_1 \\ r_3 := r_3 - 4r_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & -1 & a-12 & -4 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ r_3 := r_3 - r_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & a-13 & -2 & -1 & 1 \end{array}\right) \begin{array}{l} \\ r_2 := -r_2 \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & a-13 & -2 & -1 & 1 \end{array}\right)$$

(A) Έστω $a-13=0 \Leftrightarrow a=13$. Τότε $(K \mid B)$, με $K \in \mathbb{R}^{3\times 3}_{rref} \neq I_3$.
Άρα τότε A ιδιάζον πίνακας. Επιπλέον, $|K| = 0$, δηλ. $|A| = 0$.

(B) Έστω $a \neq 13$. Τότε $r_3 = r_3/(a-13)$, οπότε

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{a-13} & -\frac{1}{a-13} & \frac{1}{a-13} \end{array}\right) \begin{array}{l} r_1 := r_1 - 3r_3 \\ r_2 := r_2 + r_3 \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1+6/k & 3/k & -3/k \\ 0 & 1 & 0 & 2-2/k & -1-1/k & 1/D \\ 0 & 0 & 1 & -2/k & -1/k & 1/k \end{array}\right), \text{ όπου } k := \alpha - 13$$

Παράδειγμα. Μέθοδος αντιστροφής επαυξημένα πίνακα.

Έστω $A = \begin{pmatrix} 1 & a & 3 \\ 2 & b & a \\ 0 & 1 & 2 \end{pmatrix}$. Έχουμε

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & a & 3 & 1 & 0 & 0 \\ 2 & b & a & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \; r_2 := r_2 + 2r_1$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & a & 3 & 1 & 0 & 0 \\ 0 & a+2b & a+6 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \; \begin{array}{l} r_2 := r_3 \\ r_3 := r_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & a & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & a+2b & a+6 & 2 & 1 & 0 \end{array}\right) \; \begin{array}{l} r_1 := r_1 - a r_2 \\ \\ r_3 := r_3 - (b+2a) r_2 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3-2a & 1 & 0 & -a \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & a+6-(b+2a) & 2 & 1 & -b-2a \end{array}\right)$$

(A) Έστω $3a+2b=6$. Τότε $A \sim A' \in \mathbb{R}^{3\times 3}_{rref}$ με μία μηδενική γραμμή
[illegible] $|A'| = 0$)

(B) Έστω $3a+2b \neq 6$. Τότε $R_3 := R_3/(6-2b-3a)$

οπότε

$$A \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 3-2a & 1 & 0 & -a \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 6-2b-3a & 2/k & 1/k & -\frac{b+2a}{k} \end{array}\right) \begin{array}{l} r_1 := r_1 - (3-2a) r_3 \\ r_2 := r_2 - 2 r_3 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & (a-2b)/k & -(3-2a)/k & -(a^2-3b)/k \\ 0 & 1 & 0 & -4/k & 2/k & (a+b)/k \\ 0 & 0 & 1 & 2/k & 1/k & (-b-2a)/k \end{array}\right)$$

όπου $k := 6-2b-3a$.

Παράδειγμα.

Έστω $A=\begin{pmatrix} 1 & 2 & 4 \\ 2 & -1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

Έχουμε

$$(A|I_3)=\left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} r_2 := r_2 - 2r_1 \\ r_3 := r_3 - r_1 \end{array}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & -5 & -6 & -2 & 1 & 0 \\ 0 & 0 & -4 & -1 & 0 & 1 \end{array}\right) r_2 := -r_2/5$$

$$= \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 6/5 & 2/5 & -1/5 & 0 \\ 0 & 0 & -4 & -1 & 0 & 1 \end{array}\right) r_1 := r_1 - 2r_2$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 8/5 & 1/5 & 2/5 & 0 \\ 0 & 1 & 6/5 & 2/5 & -1/5 & 0 \\ 0 & 0 & -4 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} r_1 := r_1 + \frac{5}{8} r_3 \\ r_2 := r_2 + \frac{5}{6} r_3 \\ r_3 := -r_3/4 \end{array}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 8/5 & 1/5 & 2/5 & 0 \\ 0 & 1 & 6/5 & 2/5 & -1/5 & 0 \\ 0 & 0 & 1 & 1/4 & 0 & -1/4 \end{array}\right) \begin{array}{l} r_1 := r_1 - \frac{5}{8} r_3 \\ r_2 := r_2 - \frac{5}{6} r_3 \end{array}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1/5 & 2/5 & 2/5 \\ 0 & 1 & 0 & 1/10 & -5/2 & 3/10 \\ 0 & 0 & 1 & 1/4 & 0 & -1/4 \end{array}\right) = (I_3 | A^{-1})$$

Ασκήσεις.

(α) $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & -1/4 \end{pmatrix}$

(β) $\begin{pmatrix} 1 & 0 & 1 \\ -2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & -1/3 & -1/6 \\ -1/2 & 0 & 1/2 \\ 1/2 & 1/3 & 1/6 \end{pmatrix}$

(γ) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}^{-1} = \begin{pmatrix} 11/6 & -7/6 & 1/3 \\ -7/6 & 11/6 & -2/3 \\ 1/3 & -2/3 & 1/3 \end{pmatrix}$ (δ) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & -2 & 7 \\ 4 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -3/10 & -1/10 & 2/5 \\ 5/6 & 1/6 & -1/3 \\ 11/30 & 7/30 & -4/15 \end{pmatrix}$

(ε) $$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1/2 & 0 & 1 & -1 \\ -1/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Παράδειγμα. Επίλυση ΣΓΕ με τη μέθοδο του αντιστρόφου πίνακα.

$$\left\{\begin{array}{l} x+y+z=4 \\ x+2y+2z=0 \\ 2x+2y+z=-1 \end{array}\right\} \Rightarrow A\bar{x}=\bar{b}\,,\ \bar{x}=(x,y,z)^T,\ \bar{b}=(4,0,-1)^T$$

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{pmatrix}_{3\times 3}$$

$$|A|=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} \begin{array}{l} \Sigma_2 := \Sigma_2-\Sigma_1 \\ \Sigma_3 := \Sigma_3-\Sigma_1 \end{array} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix} =$$

$$= 1(-1)^{1+1}\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$$

Άρα $\bar{x}=A^{-1}\cdot\bar{b}$ (μοναδική λύση) με $A^{-1}=|A|^{-1}\,Adj(A)=|A|^{-1}\cdot(A_{ji})$

$$A_{11}=(-1)^{1+1}|M_{11}|=\begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix}=-2\,,\quad A_{12}=(-1)^{1+2}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}=3$$

$$A_{13}=(-1)^{1+3}\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}=-2\,,\quad A_{21}=(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}=1$$

$$A_{22}=(-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}=-1 \qquad A_{23}=(-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}=0$$

$$A_{31}=(-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}=-1 \qquad A_{32}=(-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=-1$$

$$A_{33}=(-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=1\,,\qquad Adj(A)=\begin{pmatrix} -2 & 3 & -2 \\ -2 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}^T$$

Ασκήσεις.

(α) $$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1/2 & 2 & 1/2 \\ 1/2 & -1 & -1/2 \\ 1/2 & -1 & -1/2 \end{pmatrix}$$

(β) $$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & & -1 \end{pmatrix}$$

(γ) $$\begin{pmatrix} 2 & 0 & 3 \\ 0 & -2 & -1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & -3/2 & -3 \\ 0 & -1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

(δ) $$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{pmatrix}^{-1} = -\frac{1}{4}\begin{pmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{pmatrix}$$

(ε) $$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 8 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 3 & -5 & 2 \\ -2 & 3 & -1 \end{pmatrix}$$

(στ) $$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 5 & -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 4/3 & -1/3 \\ 3 & -7/3 & 1/3 \end{pmatrix}$$