ΕΡΓΑΣΙΑ 5 : ΜΙΓΑΔΙΚΕΣ ΣΥΝΑΡΤΗΣΕΙΣ

1. Να αποδείξετε τις ιδιότητες των παραγράφων 5.2.3 και 5.2.6

Παρ. 5.2.3

i) $\boxed{e^{z_1}\cdot e^{z_2}=e^{z_1+z_2}}$

Έστω $z_1=x_1+iy_1$, $z_2=x_2+iy_2$. Τότε : $z_1+z_2=(x_1+x_2)+i(y_1+y_2)$

Οπότε : $e^{z_1+z_2}\overset{(*)}{=}e^{x_1+x_2}\left[\cos(y_1+y_2)+i\sin(y_1+y_2)\right]$ (1) και

$$e^{z_1}\cdot e^{z_2}\overset{(*)}{=}e^{x_1}(\cos y_1+i\sin y_1)\cdot e^{x_2}(\cos y_2+i\sin y_2)$$
$$=e^{x_1+x_2}(\cos y_1+i\sin y_1)\cdot(\cos y_2+i\sin y_2)$$
$$=e^{x_1+x_2}(\cos y_1\cdot\cos y_2+i\sin y_2\cos y_1+i\sin y_1\cos y_2+i^2\sin y_1\sin y_2)$$
$$=e^{x_1+x_2}(\cos y_1\cdot\cos y_2+i\sin y_2\cos y_1+i\sin y_1\cos y_2-\sin y_1\sin y_2)$$
$$=e^{x_1+x_2}\left[(\cos y_1\cdot\cos y_2-\sin y_1\sin y_2)+i(\sin y_2\cos y_1+\sin y_1\cos y_2)\right]$$
$$=e^{x_1+x_2}\left[\cos(y_1+y_2)+i\sin(y_1+y_2)\right]$$
$$\overset{(1)}{=}e^{z_1+z_2}$$

Σημ : Ισχύει ότι $e^z=e^x(\cos y+i\sin y),\alpha\nu\ z=x+iy\,(*)$

ii) $\boxed{e^z\neq 0\,\gamma\iota\alpha\,\kappa\alpha\theta\varepsilon\,z\in\mathbb{C}}$

$A\nu\ z=x+iy\ \tau o\tau\varepsilon\, e^z=e^x(\cos y+i\sin y)$

Έστω ότι:

$$e^z=0\Leftrightarrow e^x(\cos y+i\sin y)=0$$
$$\Leftrightarrow\begin{cases}e^x=0\,(\alpha\delta\upsilon\nu\alpha\tau\eta,\alpha\phi o\upsilon\, e^x>0)\\ \cos y+i\sin y=0+i0\Rightarrow\begin{cases}\cos y=0\\ \kappa\alpha\iota\\ \sin y=0\end{cases}\text{ΑΤΟΠΟ}\,\alpha\phi o\upsilon\sin^2 y+\cos^2 y=1\end{cases}$$

Άρα $e^z\neq 0\,\gamma\iota\alpha\,\kappa\alpha\theta\varepsilon\,z\in\mathbb{C}$

iii) $\boxed{e^z=1\Rightarrow z=2\kappa\pi i}$

Έστω $z=x+iy$. Τότε :

$$e^z=1\Rightarrow e^x(\cos y+i\sin y)=1$$
$$\Rightarrow e^x(\cos y+i\sin y)=1+0i$$
$$\Rightarrow e^x\cos y+ie^x\sin y=1+0i$$
$$\Rightarrow\begin{cases}e^x\cos y=1\ (1)\\ \kappa\alpha\iota\\ e^x\sin y=0\overset{e^x\neq 0}{\Rightarrow}\sin y=0\Rightarrow y=2k\pi\ \eta\ y=2k\pi+\pi\end{cases}$$

$$A\nu\ \ y=2k\pi+\pi\Rightarrow\cos y=-1\ \kappa\alpha\iota\ \alpha\pi o\,(1)\Rightarrow e^x=-1\,A\delta\upsilon\nu\alpha\tau\eta\,(e^x>0)$$
$$A\nu\ \ y=2k\pi\Rightarrow\cos y=1\,\alpha\pi o\,(1)\Rightarrow e^x=1\Rightarrow x=0$$
$$A\rho\alpha\ x=0,y=2k\pi\Rightarrow z=0+i2k\pi\Rightarrow z=2k\pi i$$

Παρ. 5.2.6

i) $\boxed{a^{z_1}\cdot a^{z_2}=a^{z_1+z_2}}$

$$\left.\begin{aligned}a^{z_1}&=e^{z_1\ln a}\\ a^{z_2}&=e^{z_2\ln a}\end{aligned}\right\}\Rightarrow a^{z_1}\cdot a^{z_2}=e^{z_1\ln a}\cdot e^{z_2\ln a}=e^{z_1\ln a+z_2\ln a}=e^{(z_1+z_2)\ln a}=a^{z_1+z_2}$$

ii) $\boxed{\left(a^{z_1}\right)^{z_2}=a^{z_1z_2}}$

$$a^{z_1z_2}=e^{z_1z_2\ln a}=\left(e^{z_1\ln a}\right)^{z_2}=\left(a^{z_1}\right)^{z_2}$$

2. Δείξτε ότι :

i) $\boxed{\sin^2 z+\cos^2 z=1}$

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i},\cos z=\frac{e^{iz}+e^{-iz}}{2}$$

Άρα:

$$\sin^2 z+\cos^2 z=\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2+\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2=\frac{\left(e^{iz}-e^{-iz}\right)^2}{-4}+\frac{\left(e^{iz}+e^{-iz}\right)^2}{4}=$$

$$=\frac{\left(e^{iz}+e^{-iz}\right)^2-\left(e^{iz}-e^{-iz}\right)^2}{4}=$$

$$=\frac{\left[\left(e^{iz}+e^{-iz}\right)-\left(e^{iz}-e^{-iz}\right)\right]\left[\left(e^{iz}+e^{-iz}\right)+\left(e^{iz}-e^{-iz}\right)\right]}{4}=$$

$$=\frac{\left(\cancel{e^{iz}}+e^{-iz}-\cancel{e^{iz}}+e^{-iz}\right)\left(e^{iz}+\cancel{e^{-iz}}+e^{iz}-\cancel{e^{-iz}}\right)}{4}=$$

$$=\frac{2e^{-iz}\cdot 2e^{iz}}{4}=\frac{4}{4}=1$$

ii) $\boxed{\sin(-z)=-\sin z}$

$$\sin(-z)=\frac{e^{-iz}-e^{iz}}{2i}=-\frac{e^{iz}-e^{-iz}}{2i}=-\sin z$$

iii) $\boxed{\sin(z_1+z_2)=\sin z_1\cos z_2+\cos z_1\sin z_2}$

$$\sin z_1\cos z_2+\cos z_1\sin z_2=\frac{e^{iz_1}-e^{-iz_1}}{2i}\cdot\frac{e^{iz_2}+e^{-iz_2}}{2}+\frac{e^{iz_1}+e^{-iz_1}}{2}\cdot\frac{e^{iz_2}-e^{-iz_2}}{2i}=$$

$$=\frac{e^{i(z_1+z_2)}+\cancel{e^{i(z_1-z_2)}}-\cancel{e^{-i(z_1-z_2)}}-e^{-i(z_1+z_2)}}{4i}+\frac{e^{i(z_1+z_2)}-\cancel{e^{i(z_1-z_2)}}+\cancel{e^{-i(z_1-z_2)}}-e^{-i(z_1+z_2)}}{4i}=$$

$$=\frac{2e^{i(z_1+z_2)}-2e^{-i(z_1+z_2)}}{4i}=\frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i}=\sin(z_1+z_2)$$

3. Όμοια ότι : $\boxed{\tan^{-1} z = \frac{1}{2i} Ln\left(\frac{1+iz}{1-iz}\right)}$

Έστω

$$w = \tan z = \frac{\sin z}{\cos z} = \frac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} = \frac{\not{2}\left(e^{iz} - e^{-iz}\right)}{\not{2}i\left(e^{iz} + e^{-iz}\right)} \Leftrightarrow$$

$$w = \frac{e^{iz} - \frac{1}{e^{iz}}}{i\left(e^{iz} + \frac{1}{e^{iz}}\right)} = \frac{e^{2iz} - 1}{i\left(e^{2iz} + 1\right)} \Leftrightarrow iw\left(e^{2iz} + 1\right) = e^{2iz} - 1 \Leftrightarrow iwe^{2iz} + iw = e^{2iz} - 1 \Leftrightarrow$$

$$iwe^{2iz} - e^{2iz} = -1 - iw \Leftrightarrow e^{2iz}\left(iw - 1\right) = -1 - iw \Leftrightarrow e^{2iz} = \frac{-1 - iw}{iw - 1} \Leftrightarrow$$

$$e^{2iz} = \frac{1+iw}{1-iw} \Leftrightarrow 2iz = Ln\left(\frac{1+iw}{1-iw}\right) \Leftrightarrow z = \frac{1}{2i} \cdot Ln\left(\frac{1+iw}{1-iw}\right)$$

Άρα : $\tan^{-1} z = \frac{1}{2i} Ln\left(\frac{1+iz}{1-iz}\right)$

4. Δείξτε ότι :

i) $\boxed{\cosh^2 z - \sinh^2 z = 1}$

$$\sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}$$

Άρα:

$$\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 =$$

$$= \frac{\left(e^z + e^{-z}\right)^2 - \left(e^z - e^{-z}\right)^2}{4} =$$

$$= \frac{\left[\left(e^z + e^{-z}\right) - \left(e^z - e^{-z}\right)\right]\left[\left(e^z + e^{-z}\right) + \left(e^z - e^{-z}\right)\right]}{4} =$$

$$= \frac{\left(\not{e^z} + e^{-z} - \not{e^z} + e^{-z}\right)\left(e^z + \not{e^{-z}} + e^z - \not{e^{-z}}\right)}{4} =$$

$$= \frac{2e^{-z} \cdot 2e^z}{4} = \frac{4}{4} = 1$$

ii) $\boxed{\sinh(-z) = -\sinh z}$

$$\sinh(-z) = \frac{e^{-z} - e^z}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh z$$

iii) $\boxed{\sinh(z_1 - z_2) = \sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2}$

$$\sinh z_1 \cosh z_2 - \cosh z_1 \sinh z_2 = \frac{e^{z_1} - e^{-z_1}}{2} \cdot \frac{e^{z_2} + e^{-z_2}}{2} - \frac{e^{z_1} + e^{-z_1}}{2} \cdot \frac{e^{z_2} - e^{-z_2}}{2} =$$

$$= \frac{\cancel{e^{(z_1+z_2)}} + e^{(z_1-z_2)} - e^{-(z_1-z_2)} - \cancel{e^{-(z_1+z_2)}}}{4} - \frac{\cancel{e^{(z_1+z_2)}} - e^{(z_1-z_2)} + e^{-(z_1-z_2)} - \cancel{e^{-(z_1+z_2)}}}{4} =$$

$$= \frac{2e^{(z_1-z_2)} - 2e^{-(z_1-z_2)}}{4} = \frac{e^{(z_1-z_2)} - e^{-(z_1-z_2)}}{2} = \sinh(z_1 - z_2)$$

5. Όμοια ότι : $\boxed{\begin{array}{l} \sinh^{-1} z = Ln\left(z + \sqrt{z^2+1}\right) \\ \tanh^{-1} z = \frac{1}{2} Ln\left(\frac{1+z}{1-z}\right) \end{array}}$ **με κατάλληλους περιορισμούς στο z**

Έστω :

$$w = \sinh z = \frac{e^z - e^{-z}}{2} \Leftrightarrow$$

$$2w = e^z - e^{-z} \Leftrightarrow 2w = e^z - \frac{1}{e^z} \Leftrightarrow 2w = \frac{\left(e^z\right)^2 - 1}{e^z} \Leftrightarrow$$

$$\Leftrightarrow \left(e^z\right)^2 - 1 = 2we^z \Leftrightarrow \left(e^z\right)^2 - 2we^z - 1 = 0$$

$$\Delta = 4w^2 + 4 = 4(w^2+1)$$

$$A\rho\alpha : e^z = \frac{2w \pm 2\sqrt{w^2+1}}{2} = w \pm \sqrt{w^2+1}$$

Επειδή κάθε μιγαδικός αριθμός έχει δύο αντίθετες τετραγωνικές ρίζες, χωρίς βλάβη της γενικότητας κρατάω τη μία λύση της παραπάνω εξίσωσης. Άρα :

$$e^z = w + \sqrt{w^2+1} \Leftrightarrow z = Ln\left(w + \sqrt{w^2+1}\right)$$

Άρα : $\boxed{\sinh^{-1} z = Ln\left(z + \sqrt{z^2+1}\right)}$

Έστω

$$w = \tanh z = \frac{\sinh z}{\cosh z} = \frac{\frac{e^z - e^{-z}}{2}}{\frac{e^z + e^{-z}}{2}} = \frac{\cancel{2}\left(e^z - e^{-z}\right)}{\cancel{2}\left(e^z + e^{-z}\right)} \Leftrightarrow$$

$$w = \frac{e^z - \frac{1}{e^z}}{e^z + \frac{1}{e^z}} = \frac{e^{2z} - 1}{e^{2z} + 1} \Leftrightarrow e^{2z} - 1 = w\left(e^{2z} + 1\right) \Leftrightarrow$$

$$e^{2z}(1-w) = 1 + w \Leftrightarrow e^{2z} = \frac{1+w}{1-w} \Leftrightarrow$$

$$2z = Ln\left(\frac{1+w}{1-w}\right) \Leftrightarrow z = \frac{1}{2} Ln\left(\frac{1+w}{1-w}\right)$$

Δηλ.

$$\boxed{\tanh^{-1} z = \frac{1}{2} Ln\left(\frac{1+z}{1-z}\right)}$$

6. Δείξτε ότι οι sinz, tanz και cotz είναι περιττές συναρτήσεις, ενώ η cosz άρτια συνάρτηση.

- sinz περιττή γιατί: $\sin(-z) = -\sin z$ (ασκ. 2ii), δηλ. f(-z)=-f(z)
- tanz περιττή γιατί:

$$\tan z = \frac{\sin z}{\cos z} = \frac{\dfrac{e^{iz} - e^{-iz}}{2i}}{\dfrac{e^{iz} + e^{-iz}}{2}} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\tan(-z) = \frac{e^{-iz} - e^{iz}}{i(e^{-iz} + e^{iz})} = \frac{-(e^{iz} - e^{-iz})}{i(e^{iz} + e^{-iz})} = -\frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = -\tan z$$

δηλ. f(-z)=-f(z)

- cotz περιττή γιατί:

$$\cot z = \frac{\cos z}{\sin z} = \frac{\dfrac{e^{iz} + e^{-iz}}{2}}{\dfrac{e^{iz} - e^{-iz}}{2i}} = \frac{i(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})}$$

$$\cot(-z) = \frac{i(e^{-iz} + e^{iz})}{(e^{-iz} - e^{iz})} = \frac{i(e^{-iz} + e^{iz})}{-(e^{iz} - e^{-iz})} = -\frac{i(e^{-iz} + e^{iz})}{e^{iz} - e^{-iz}} = -\cot z$$

δηλ. f(-z)=-f(z)

- cosz άρτια γιατί:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\cos(-z) = \frac{e^{-iz} + e^{-i(-z)}}{2} = \frac{e^{-iz} + e^{iz}}{2} = \cos z$$

Δηλ. f(-z)=f(z)

7. **Να υπολογιστούν τα** $\sin^{-1} 2, \cos^{-1} i \ \kappa\alpha\iota \ \sinh^{-1} i$

- $\sin^{-1} z = \frac{1}{i} Ln\left(iz + \sqrt{1-z^2}\right)$

  Άρα :

  $$\sin^{-1} 2 \ = \frac{1}{i} Ln\left(i2 + \sqrt{1-2^2}\right) =$$
  $$= \frac{1}{i} Ln\left(2i + \sqrt{3i^2}\right) =$$
  $$= \frac{1}{i} Ln\left(2i + i\sqrt{3}\right) =$$
  $$= \frac{1}{i} Ln\left[\left(2+\sqrt{3}\right)i\right] \ (1)$$

  Έστω $z = 0 + \left(2+\sqrt{3}\right)i$. Τότε :

  $|z| = 2+\sqrt{3}$

  $$\left.\begin{array}{l} \cos\theta = 0 \\ \sin\theta = \dfrac{2+\sqrt{3}}{2+\sqrt{3}} = 1 \end{array}\right\} \Rightarrow \theta = \frac{\pi}{2}$$

  Άρα : $Ln\left[\left(2+\sqrt{3}\right)i\right] = \ln\left(2+\sqrt{3}\right) + i\frac{\pi}{2}$ και η (1) γίνεται :

  $$\sin^{-1} 2 \ = \frac{1}{i}\left[ln\left(2+\sqrt{3}\right) + i\frac{\pi}{2}\right] \Rightarrow$$
  $$\sin^{-1} 2 = \frac{ln\left(2+\sqrt{3}\right)}{i} + \frac{\pi}{2} \Rightarrow$$
  $$\sin^{-1} 2 = \frac{iln\left(2+\sqrt{3}\right)}{i^2} + \frac{\pi}{2} \Rightarrow$$
  $$\sin^{-1} 2 = \frac{\pi}{2} - iln\left(2+\sqrt{3}\right)$$

- $\cos^{-1} z = \frac{1}{i} Ln\left(z + \sqrt{z^2 - 1}\right)$

  Άρα :

  $$\cos^{-1} i = \frac{1}{i} Ln\left(i + \sqrt{i^2-1}\right) = \frac{1}{i} Ln\left(i + \sqrt{2i^2}\right) = \frac{1}{i} Ln\left(i + i\sqrt{2}\right) = \frac{1}{i} Ln\left[\left(1+\sqrt{2}\right)i\right] (2)$$

  Έστω $z = 0 + \left(1+\sqrt{2}\right)i$. Τότε : $|z| = 1+\sqrt{2}$ και :

  $$\left.\begin{array}{l} \cos\theta = 0 \\ \sin\theta = \dfrac{1+\sqrt{2}}{1+\sqrt{2}} = 1 \end{array}\right\} \Rightarrow \theta = \frac{\pi}{2}$$

  Άρα $Ln\left[\left(1+\sqrt{2}\right)i\right] = \ln\left(1+\sqrt{2}\right) + i\frac{\pi}{2}$ και η (2) γίνεται :

$$\cos^{-1} i = \frac{1}{i}\left[\ln\left(1+\sqrt{2}\right)+i\frac{\pi}{2}\right] = \frac{i}{i^2}\left[\ln\left(1+\sqrt{2}\right)+i\frac{\pi}{2}\right] = -i\left[\ln\left(1+\sqrt{2}\right)+i\frac{\pi}{2}\right] \Rightarrow$$

$$\cos^{-1} i = -i\ln\left(1+\sqrt{2}\right) - i^2\frac{\pi}{2} \Rightarrow \cos^{-1} i = \frac{\pi}{2} - i\ln\left(1+\sqrt{2}\right)$$

- $\sinh^{-1} z = Ln\left(z+\sqrt{z^2+1}\right)$

Άρα $\sinh^{-1} i = Ln\left(i+\sqrt{i^2+1}\right) = Ln\left(i+\sqrt{-1+1}\right) = Lni$ (3)

Έστω $z = 0+1i$ τότε $|z| = 1$ και

$$\left.\begin{array}{l}\cos\theta = 0\\ \sin\theta = 1\end{array}\right\} \Rightarrow \theta = \frac{\pi}{2}$$

Άρα $Lni = \ln 1 + i\frac{\pi}{2} = 0 + i\frac{\pi}{2} = i\frac{\pi}{2}$ και η (3) γίνεται :

$$\sinh^{-1} i = i\frac{\pi}{2}$$

8. **Δείξτε ότι** $\overline{\sin z} = \sin\bar{z}, \overline{\cos z} = \cos\bar{z} \; \kappa\alpha\iota \; \overline{\tan z} = \tan\bar{z}$ **για κάθε** $z \in \mathbb{C}$

- $\boxed{\overline{\sin z} = \sin\bar{z}}$

Έστω $z = x+iy$ τότε $\bar{z} = x-iy$ και :

$$\sin\bar{z} = \frac{e^{i\bar{z}} - e^{-i\bar{z}}}{2i} = \frac{e^{i(x-yi)} - e^{-i(x-yi)}}{2i} = \frac{e^{ix+y} - e^{-ix-y}}{2i}$$

$$\Rightarrow \sin\bar{z} = \frac{e^{y+ix} - e^{-y-ix}}{2i}$$

$$\Rightarrow \sin\bar{z} = \frac{e^{y}\left(\cos x + i\sin x\right) - e^{-y}\left[\cos(-x) + i\sin(-x)\right]}{2i} \qquad \left[\alpha\nu \;\; z = x+yi \;\; \tau o\tau\varepsilon \;\; e^{z} = e^{x}(\cos y + i\sin y)\right]$$

$$\boxed{\Rightarrow \sin\bar{z} = \frac{e^{y}\left(\cos x + i\sin x\right) - e^{-y}\left(\cos x - i\sin x\right)}{2i}} \quad (1)$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+yi)} - e^{-i(x+yi)}}{2i} = \frac{e^{ix-y} - e^{-ix+y}}{2i}$$

$$\Rightarrow \sin z = \frac{e^{-y}(\cos x + i\sin x) - e^{y}\left[\cos(-x) + i\sin(-x)\right]}{2i}$$

$$\Rightarrow \sin z = \frac{e^{-y}(\cos x + i\sin x) - e^{y}\left(\cos x - i\sin x\right)}{2i}$$

Άρα :

$$\overline{\sin z} = \frac{\overline{e^{-y}(\cos x + i\sin x) - e^{y}\left(\cos x - i\sin x\right)}}{\overline{2i}} \qquad \left[\mathrm{I}\sigma\chi\upsilon\varepsilon\iota \;\; \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}\right]$$

$$\Rightarrow \overline{\sin z} = \frac{\overline{e^{-y}\cos x + ie^{-y}\sin x - e^{y}\cos x + ie^{y}\sin x}}{-2i}$$

$$\Rightarrow \overline{\sin z} = \frac{\overline{\left(e^{-y}\cos x - e^{y}\cos x\right) + i\left(e^{-y}\sin x + e^{y}\sin x\right)}}{-2i}$$

$$\Rightarrow \overline{\sin z} = \frac{\left(e^{-y}\cos x - e^{y}\cos x\right) - i\left(e^{-y}\sin x + e^{y}\sin x\right)}{-2i}$$

$$\Rightarrow \overline{\sin z} = \frac{-\left(e^{-y}\cos x - e^{y}\cos x\right) + i\left(e^{-y}\sin x + e^{y}\sin x\right)}{2i}$$

$$\Rightarrow \overline{\sin z} = \frac{e^{y}\cos x - e^{-y}\cos x + ie^{-y}\sin x + ie^{y}\sin x}{2i}$$

$$\Rightarrow \overline{\sin z} = \frac{e^{y}\cos x + ie^{y}\sin x - e^{-y}\cos x + ie^{-y}\sin x}{2i}$$

$$\Rightarrow \overline{\sin z} = \frac{e^{y}\left(\cos x + i\sin x\right) - e^{-y}\left(\cos x - i\sin x\right)}{2i}$$

$$\boxed{\Rightarrow \overline{\sin z} = \frac{e^{y}\left(\cos x + i\sin x\right) - e^{-y}\left(\cos x - i\sin x\right)}{2i}} \quad (2)$$

Από (1) και (2) προκύπτει ότι $\boxed{\overline{\sin z} = \sin \bar{z}}$

- $\boxed{\overline{\cos z} = \cos \bar{z}}$

  Έστω $z = x + iy$ τότε $\bar{z} = x - iy$ και :

$$\cos \bar{z} = \frac{e^{i\bar{z}} + e^{-i\bar{z}}}{2} = \frac{e^{i(x-yi)} + e^{-i(x-iy)}}{2} \Rightarrow$$

$$\cos \bar{z} = \frac{e^{ix+y} + e^{-ix-y}}{2} = \frac{e^{y+ix} + e^{-y-ix}}{2} \Rightarrow$$

$$\cos \bar{z} = \frac{1}{2}\left[e^{y}(\cos x + i\sin x) + e^{-y}\left(\cos(-x) + i\sin(-x)\right)\right] \overset{\cos(-x)=\cos x}{\underset{\sin(-x)=-\sin x}{\Rightarrow}}$$

$$\cos \bar{z} = \frac{1}{2}\left[e^{y}(\cos x + i\sin x) + e^{-y}\left(\cos x - i\sin x\right)\right]$$

$$\boxed{\cos \bar{z} = \frac{1}{2}\left[e^{y}(\cos x + i\sin x) + e^{-y}\left(\cos x - i\sin x\right)\right]} \quad (3)$$

Όμως :

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \Rightarrow$$

$$\cos z = \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{1}{2}\left(e^{-y+ix} + e^{y-ix}\right) \Rightarrow$$

$$\cos z = \frac{1}{2}\left[e^{-y}\left(\cos x + i\sin x\right) + e^{y}\left(\cos(-x) + i\sin(-x)\right)\right] \Rightarrow$$

$$\cos z = \frac{1}{2}\left[e^{-y}\left(\cos x + i\sin x\right) + e^{y}\left(\cos x - i\sin x\right)\right] \Rightarrow$$

$$\cos z = \frac{1}{2}\left[e^{-y}\cos x + ie^{-y}\sin x + e^{y}\cos x - ie^{y}\sin x\right] \Rightarrow$$

$$\cos z = \frac{1}{2}\left[\left(e^{-y}\cos x + e^{y}\cos x\right) + i\left(e^{-y}\sin x - e^{y}\sin x\right)\right]$$

Άρα :

$$\overline{\cos z} = \frac{1}{2}\left[\left(e^{-y}\cos x + e^{y}\cos x\right) - i\left(e^{-y}\sin x - e^{y}\sin x\right)\right] \Rightarrow$$

$$\overline{\cos z} = \frac{1}{2}\left[e^{-y}\cos x + e^{y}\cos x - ie^{-y}\sin x + ie^{y}\sin x\right] \Rightarrow$$

$$\boxed{\overline{\cos z} = \frac{1}{2}\left[e^{y}(\cos x + i\sin x) + e^{-y}\left(\cos x - i\sin x\right)\right]} \quad (4)$$

Από (3) και (4) προκύπτει ότι $\boxed{\overline{\cos z} = \cos\bar{z}}$

- $\boxed{\overline{\tan z} = \tan\bar{z}}$

  ‘Εχω δείξει ότι : $\overline{\sin z} = \sin\bar{z}, \overline{\cos z} = \cos\bar{z}$ (*). Άρα :

  $$\tan\bar{z} = \frac{\sin\bar{z}}{\cos\bar{z}} \overset{(*)}{=} \frac{\overline{\sin z}}{\overline{\cos z}} = \overline{\left(\frac{\sin z}{\cos z}\right)} = \overline{\tan z} \qquad \left[\alpha\varphi o\upsilon \ \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}\right]$$